Problem 2.19

Consider the projectile of Section 2.3. **(a)** Assuming there is no air resistance, write down the position (x, y) as a function of t, and eliminate t to give the trajectory y as a function of x. **(b)** The correct trajectory, including a linear drag force, is given by (2.37). Show that this reduces to your answer for part (a) when air resistance is switched off (τ and $v_{\text{ter}} = g\tau$ both approach infinity). [*Hint:* Remember the Taylor series (2.40) for $\ln(1-\epsilon)$.]

Solution

Part (a)

Newton's second law gives two equations of motion, one for each dimension the projectile moves in.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \end{cases}$$

Since there's no air resistance, the only force to consider is the one due to gravity.

$$\begin{cases} 0 = ma_x \\ -mg = ma_y \end{cases}$$

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_{x\text{o}} \\ \dfrac{dy}{dt} = -gt + v_{y\text{o}} \end{cases}$$

$$\begin{cases} x(t) = v_{x\text{o}}t + x_\text{o} \\ y(t) = -\dfrac{1}{2}gt^2 + v_{y\text{o}}t + y_\text{o} \end{cases}$$

Take the launching site to be the origin so that $x_o = 0$ and $y_o = 0$.

$$\begin{cases} x(t) = v_{xo}t \\ \\ y(t) = -\frac{1}{2}gt^2 + v_{yo}t \end{cases}$$

Solve the first equation for t,

$$t = \frac{x}{v_{xo}},$$

and plug it into the second equation.

$$y = -\frac{1}{2}g\left(\frac{x}{v_{xo}}\right)^2 + v_{yo}\left(\frac{x}{v_{xo}}\right)$$

$$y(x) = -\frac{gx^2}{2v_{xo}^2} + \frac{v_{yo}x}{v_{xo}}$$

This is the graph of a parabola that opens downward.

Part (b)

Equation (2.37) is on page 54 and gives the trajectory of a particle in a medium with linear air resistance $\mathbf{f} = -b\mathbf{v}$.

$$y = \frac{v_{yo} + v_{ter}}{v_{xo}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right) \tag{2.37}$$

$$y(x) = \frac{v_{yo} + \frac{mg}{b}}{v_{xo}}x + \frac{mg}{b}\frac{m}{b}\ln\left(1 - \frac{bx}{mv_{xo}}\right)$$

When air resistance is minimal, $b \approx 0$, and $bx/(mv_{xo})$ is very small. Consequently, a Taylor series expansion of $\ln(1-\epsilon)$ about $\epsilon = 0$ is justified. This expansion is in Equation (2.40) on page 55.

$$y(x) = \frac{v_{yo} + \frac{mg}{b}}{v_{xo}}x + \frac{mg}{b}\frac{m}{b}\left[-\left(\frac{bx}{mv_{xo}}\right) - \frac{1}{2}\left(\frac{bx}{mv_{xo}}\right)^2 - \frac{1}{3}\left(\frac{bx}{mv_{xo}}\right)^3 - \cdots\right]$$

Since we want this to simplify to the result in part (a), we keep only the linear and quadratic terms and argue that all higher-order terms are negligible compared to the first two.

$$\begin{aligned} y(x) &\approx \frac{v_{yo} + \frac{mg}{b}}{v_{xo}}x + \frac{mg}{b}\frac{m}{b}\left[-\left(\frac{bx}{mv_{xo}}\right) - \frac{1}{2}\left(\frac{bx}{mv_{xo}}\right)^2\right] \\ &\approx \frac{v_{yo} + \frac{mg}{b}}{v_{xo}}x - \frac{mgx}{bv_{xo}} - \frac{gx^2}{2v_{xo}^2} \\ &\approx \frac{v_{yo}}{v_{xo}}x - \frac{gx^2}{2v_{xo}^2} \end{aligned}$$